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DYNAMICS OF SQUEEZING FROM GENERALIZED COHERENT STATES

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Abstract

We extend the definition of generalized coherent states to include the case of time-dependent dispersion. We introduce a suitable operator providing displacement and dynamical rescaling from an arbitrary ground state. As a consequence, squeezing is naturally embedded in this framework, and its dynamics is ruled by the evolution equation for the dispersion. Our construction provides a displacement-operator method to obtain the squeezed states of arbitrary systems.

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Introduction. Coherent states are the quantum states that are closest to a classical, localized time-evolution; after the pioneering work by Schrödinger [1], they were discovered and their structure thoroughly clarified in the modern language of quantum field theory by Glauber, Klauder, and Sudarshan [2], [3].

Besides their conceptual relevance to the understanding of basic features of quantum mechanics, they are by now an indispensable mathematical tool in many fields of theoretical physics, ranging from quantum field theory to statistical mechanics [4], [5].

Their extension, squeezed states [6], have come to play an increasingly important role in the last decade; beyond the traditional context of quantum electrodynamics and quantum optics [7], they also appear to be of special interest in the theory of quantum nondemolition measurements applied to gravitational wave detection [8].

In this note we address the problem of constructing and deriving the dynamical properties of squeezed states for arbitrary systems.

The results that we are going to present stem from a new approach to coherent states that is based on Nelson stochastic mechanics, a quantization scheme [9], whose motivation arises from a cross-breeding of ideas and methods of euclidean quantum field theory [10], the theory of stochastic differential equations [11], and the theory of stochastic optimal control [12].

New results and insights can then be obtained by looking at quantum coherence in terms of general properties of classical diffusion processes. This approach has been carried out gradually.

We first derived the standard harmonic-oscillator coherent and squeezed states as the Nelson diffusions minimizing the osmotic uncertainty relations of stochastic mechanics [13].

Next, we introduced a class of generalized coherent states as the Nelson diffusions with classical current velocity and wave-like propagating osmotic velocity [14]. These coherent states follow a classical motion in generic time-dependent potentials without spreading of the wave packet. The evolution is controlled by a feedback mechanism that allows the packets to remain coherent through a continuous dynamical readjustment.

The connection with standard operatorial languages was then provided by showing that the new class of states is generated letting the Glauber displacement operator $\hat{D}(\alpha)$ act on the ground states of arbitrary potentials in the coordinate representation [15]. In this way we derived a complete dynamical

description of the displacement-operator generalized coherent states.

In this letter we extend the scheme previously developed to include the case of time-dependent dispersion Δq. By resorting to the stochastic framework we are able to derive the desired evolution equation for Δq, as well as the general form of the wave functions associated to this new class of coherent states.

A suitable displacement operator is then introduced in the coordinate representation in order to obtain these states in standard operatorial language: they are displacement-operator generalized coherent states with time-dependent dispersion.

In fact, the displacement operator acts as the product of two distinct mappings: the ordinary Glauber displacement operator $\hat{D}(\alpha)$ and a dynamical rescaling operator, namely a dynamical "squeeze" operator.

Squeezing is then naturally embedded in this scheme, and the evolution equation for Δq yields also the dynamical equation controlling the time-evolution of squeezing.

At the same time, the above construction provides a natural extension of the displacement-operator method to define a class of squeezed states for arbitrary potentials.

Stochastic mechanics. We shall quickly review the basic ingredients of the stochastic formulation of quantum mechanics that will be needed in the following.

This quantization procedure rests on two basic prescriptions; the first one, kinematical, promotes the configuration of a classical system to a conservative diffusion process with diffusion coefficient equal to $\hbar/2m$.

If we denote by $q(t)$ the configurational variable for a point particle with mass m, this prescription reads

$$dq(t) = v_{(+)}(q(t), t)dt + \sqrt{\frac{\hbar}{2m}}dw(t) \, , \quad dt > 0 \, . \tag{1}$$

In the above stochastic differential equation $v_{(+)}$ is a (forward) drift field that is determined by assigning the dynamics, and w is the standard Wiener process.

An intuitive manner to look at Eq. (1) is to consider it as the appropriate quantum form of the classical kinematical prescription: the Wiener process models quantum fluctuations, just as in the theory of beams dynamics in particle accelerators.

Under very general mathematical conditions, the diffusion $q(t)$ admits the backward representation

$$dq(t) = v_{(-)}(q(t),t)dt + \sqrt{\frac{\hbar}{2m}}dw^{*}(t)\,, \qquad dt > 0\ , \tag{2}$$

where w^{*} is a time-reversed Wiener process and the backward drift is defined by the relation

$$v_{(-)} \;=\; v_{(+)} \;-\; \frac{\hbar}{2m}\nabla \ln \rho\,. \tag{3}$$

In Eq.(3) above $\rho(x,t)$ denotes the probability density associated to the process $q(t)$. It is useful to introduce the hydrodynamic representation in terms of the osmotic and current velocity, defined respectively by

$$u(x,t) = \frac{v_{(+)} - v_{(-)}}{2}\,, \tag{4}$$

and

$$v(x,t) = \frac{v_{(+)} + v_{(-)}}{2}\,. \tag{5}$$

In the hydrodyanmic picture, the Fokker-Planck equation associated to Eqs. (1)-(2) takes the form of a continuity equation:

$$\partial_t \rho \;=\; -\nabla(\rho v)\,. \tag{6}$$

The dynamical prescription is introduced by defining the mean regularized classical action A. In the hydrodynamic Eulerian picture it is a functional of the couple (ρ, v):

$$A \;=\; \int_{t_a}^{t_b} \left[\frac{m}{2}(v^2 - u^2) - \Phi\right]\rho d^3x\,, \tag{7}$$

where $\Phi(x,t)$ denotes the external potential.

The equations of motion are then obtained by extremizing A against smooth variations $\delta\rho$, δv vanishing at the boundaries of integration, with the continuity equation taken as a constraint.

After standard calculations one obtains

$$\partial_t v + (v\cdot\nabla)v - \frac{\hbar^2}{4m^2}\nabla\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) = -\nabla\Phi\,, \tag{8}$$

with the current velocity fixed to be a gradient field at all points where $\rho > 0$: $v = \nabla S/m$, where $S(x,t)$ is a scalar function.

Defining the wave function $\Psi(x,t)$ for a generic quantum state in the hydrodynamic form $\Psi = \sqrt{\rho}\exp[iS/\hbar]$, one immediately has that the continuity equation together with the dynamical equation Eq. (8) are equivalent to the Schrödinger equation.

The space integral of Eq. (8) yields the Hamilton-Jacobi-Madelung equation. It is useful for what follows to write this equation in the form

$$\partial_t S + \frac{m}{2}v^2 - \frac{m}{2}u^2 - \frac{\hbar}{2}\partial_x u \; = \; -\Phi \, . \tag{9}$$

The correspondence between expectations and correlations defined in the stochastic and in the canonic pictures are

$$\langle \hat{q} \rangle = E(q) \, , \qquad \langle \hat{p} \rangle = mE(v) \, ,$$
$$\tag{10}$$
$$\Delta\hat{q} = \Delta q \, , \qquad (\Delta\hat{p})^2 = m^2[(\Delta u)^2 + (\Delta v)^2] \, ,$$

The following chain inequality holds:

$$(\Delta\hat{q})^2(\Delta\hat{p})^2 \; \geq \; m^2(\Delta q)^2(\Delta u)^2 \; \geq \; \frac{\hbar^2}{4} \, . \tag{11}$$

In the above relations $\hat{q}$ and $\hat{p}$ denote the position and momentum observables in the Schrödinger picture, $\langle \cdot \rangle$ denotes the expectation value of the operators in the given state Ψ, $E(\cdot)$ is the expectation value of the stochastic variables associated in the Nelson picture to the state $\{\rho, v\}$, and $\Delta(\cdot)$ denotes the root mean square deviation.

The inequalities Eq. (11), i.e. the osmotic uncertainty relation and its equivalence with the momentum-position uncertainty were proven in Ref.[16].

Harmonic-oscillator coherent and squeezed states. Saturation of the osmotic uncertainty relation Eq. (11) leads to the definition of Glauber coherent states in the stochastic picture [13]: they are Nelson diffusions of constant dispersion Δq, and with classical current velocity and linear osmotic velocity:

$$v \; = \; \langle v \rangle \qquad\qquad u \; = \; -\frac{\hbar}{2m\Delta q}\xi \, . \tag{12}$$

We have now denoted the stochastic expectations with the same symbol used for quantum ones, and we have introduced the adimensional variable $\xi = (x - \langle q \rangle)/\Delta q$.

The harmonic-oscillator squeezed states are instead Nelson diffusions with time-varying Δq, linear u of the form Eq. (12) and current velocity of the form [13]:

$$v \;=\; \langle v \rangle \,+\, \xi \frac{d}{dt} \Delta q \, . \tag{13}$$

The last term in Eq. (13) is responsible for the quantum anticommutator term appearing in the phase of the squeezed wave packets.

Of course, both the coherent and the squeezed states Eqs. (12)-(13) follow the classical motion

$$\frac{d}{dt}(m\langle v \rangle) \;=\; -\nabla \Phi(x,t)|_{x=\langle q \rangle} \, , \tag{14}$$

where, $\langle v \rangle = d\langle q \rangle/dt$, a well known classical property of quantum and stochastic expectations.

Harmonic-oscillator coherent states can also be obtained in a stochastic variational approach by extremizing the osmotic uncertainty product against smooth variations of the density ρ and of the current velocity v [17]. The possibility of extending this approach to study local minimum uncertainty behaviors in non harmonic systems is currently being investigated, see Ref. [17].

Generalized coherent states. From a dynamical point of view a coherent state is a wave packet whose centre follows a classical motion and whose dispersion is either constant or controlled in its time-evolution (squeezing).

In quantum mechanics the dynamics of mean values obeys Ehrenfest theorem: as a consequence, the coherent evolution Eq. (14) is strictly satisfied if

$$\langle \nabla \Phi(x,t) \rangle \;=\; \nabla \Phi(x,t)|_{x=\langle q \rangle} \, . \tag{15}$$

In the case of quadratic potentials the above constraint is authomatically satisfied for any quantum state.

For other generic potentials $V(x,t)$ Eq. (15) in general cannot be satisfied. However, in stochastic mechanics a particular choice of the current velocity selects an entire class of osmotic velocities, i.e. of quantum states.

In particular, we showed that, in the case of constant Δq, the choice $v = \langle v \rangle$ does not fix u to be only of the standard Glauber form Eq. (12), but

yields instead a whole class of quantum states with osmotic velocities of the wave-like propagating form $u = G(\xi)/\Delta q$, with G arbitray function [14].

What are the properties of this class of states? They are no more Heisenberg minimum uncertainty states (the latter are recovered choosing $G = -\hbar\xi/2m$). However, they obey the constraint Eq. (15) apart, at most, a constant.

Moreover, they can be obtained in the coordinate representation by applying the displacement operator on the ground state wave functions of arbitrary configurational potentials $V_0(x)$ [15].

They are then generalized coherent states of the displacement operator, following classical motion with constant dispersion; we showed that this is possible because they obey Schrödinger equation in time-dependent potentials $V(x,t)$ with a dynamical feedback mechanism allowing the wave packet to remain coherent.

These states exhaust the class of possible ones obeying a generalized Glauber condition Eq. (15). In fact, the harmonic oscillator is the trivial instance for which the feedback mechanism disappears [15].

In conclusion, we first selected via stochastic mechnanics the quantum systems, beyond the harmonic oscillator, that can obey constraint Eq. (15) and we then showed that these systems are associated with the displacement-operator coherent states, and derived their dynamical properties.

Time-dependent dispersion: generalized "dynamical" coherent states. We now proceed to build the case of time-dependent dispersion, that is we consider the more general form Eq. (13) for the current velocity of minimum uncertainty.

The generalized coherent states that we expect to select by taking the choice Eq. (13) for the current velocity would obviously be states following two coupled dynamical equations, Eq. (14) for the wave packet centre, and an evolution equation for the dispersion Δq, as in the Harmonic case.

We proceed as follows. By inserting Eq. (13) in the continuity equation we are left with the equation

$$\partial_t \rho \; = \; v \nabla \rho - \frac{1}{\Delta q} \frac{d}{dt} \Delta q \, , \tag{16}$$

whose general solution is function only of $\xi = (x - \langle q \rangle)/\Delta q$, and reminding

that ρ must be non negative it can be cast in the form

$$\rho \ = \ \exp\left[\frac{R(\xi)}{\Delta q}\right] \,, \tag{17}$$

with R any arbitrary function yielding a normalizable probability density. By Eqs. (3)-(4) one then has

$$u \ = \ \frac{1}{\Delta q} G(\xi) \,. \tag{18}$$

As a consequence, we have again that a class of osmotic velocities of wave propagating form is selected, with the arbitrary function G restricted only by the normalization requirement for ρ.

Inserting now the current velocity Eq. (13) in the equation of motion Eq. (8), by Eq. (18) one has

$$- m\xi\frac{d\Delta q}{dt} + \frac{m}{2}\nabla u^2 + \frac{\hbar}{2}\nabla^2 u \ = \ \nabla\Phi - \langle\nabla\Phi\rangle \,, \tag{19}$$

where we exploited Ehrenfest theorem $d\langle v\rangle/dt = -\langle\nabla\Phi\rangle/m$.

Letting $x = \langle q\rangle$ (i.e. $\xi = 0$) in Eq. (19), one has

$$\nabla\Phi \mid_{x=\langle q\rangle} - \langle\nabla\Phi\rangle \ = \ \frac{m}{2}\nabla u^2 \mid_{\xi=0} + \frac{\hbar}{2}\nabla^2 u \mid_{\xi=0} \ . \tag{20}$$

This is the same relation previously established in the case $v = \langle v\rangle$ (time-independent Δq); the right hand side is obviously either constant or zero except for singular potentials: in these cases u diverges in $\xi = 0$. Explicit examples and applications to non singular potentials, as well as a careful analysis of the singular cases will be discussed in detail elsewhere [18].

Next, we want to derive the phase S and the evolution equation for the dispersion Δq. This is achieved by exploiting Hamilton-Jacobi-Madelung equation, Eq. (9): reminding that v is the gradient field of S, Eq. (13) implies

$$S \ = \ m\langle v\rangle x \ + \ \frac{m}{2}\frac{(x-\langle q\rangle)^2}{\Delta q}\frac{d\Delta q}{dt} \ + \ S_0(t) \,, \tag{21}$$

where $S_0(t)$ denotes the classical phase. Inserting Eqs. (21), (13), and (18) in Eq. (9), and taking the expectation value, we obtain the evolution equation for Δq

$$\frac{m}{2}\Delta q\frac{d^2\Delta q}{dt^2} - \frac{m}{2}\langle v\rangle^2 + \frac{m}{2}\langle u^2\rangle \ = \ -\langle\Phi\rangle \,. \tag{22}$$

By Eq. (18) for u it is immediately seen that

$$\langle u^2 \rangle = \frac{K}{(\Delta q)^2} \ , \tag{23}$$

where $K = \int_{-\infty}^{\infty} G^2(\xi) d\xi$; Eq. (22) is then the desired equation for the time-evolution of the dispersion and is naturally coupled through the term in $\langle v \rangle$ with the classical equation of motion for the wave packet center $\langle q \rangle$, Eq. (14).

The general form of the wave function for such states is readily obtained putting together Eq. (17) for ρ and Eq. (21) for S:

$$\Psi(x,t) \ = \ \exp\left[\frac{R(\xi)}{\Delta q} \ + \ \frac{i}{\hbar} \left(m\langle v \rangle x + \frac{m}{2} \frac{(x - \langle q \rangle)^2}{\Delta q} \frac{d\Delta q}{dt} + S_0(t) \right) \right] . \tag{24}$$

We can rewrite this expression in more familiar terms by observing that $m\Delta q d(\Delta q)/dt = \langle \{\hat{Q}, \hat{P}\} \rangle /2$, where $\hat{Q} = \hat{q} - \langle \hat{q} \rangle$, $\hat{P} = \hat{p} - \langle \hat{p} \rangle$, and $\{,\}$ denotes the anticommutator. We thus have

$$\Psi(x,t) \ = \ \exp\left[\frac{R(\xi)}{\Delta q} \ + \ \frac{i}{\hbar} \left(\langle \hat{p} \rangle x + \frac{\langle \{\hat{Q}, \hat{P}\} \rangle}{(2\Delta q)^2} (x - \langle \hat{q} \rangle)^2 + S_0(t) \right) \right] . \tag{25}$$

The above are nonstationary states with classical motion and controlled time-dependent spreading, which evolve in the time-dependent potentials $V(x,t)$.

In the next section these generalized "dynamical" coherent states will be derived introducing Glauber displacement operator and a proper "squeeze" operator.

Displacement operator: generalized squeezed states. It is well known that generalized coherent states can be obtained extending the three different existing approaches to the definition of the harmonic-oscillator coherent states: they are, respectively, the minimum-uncertainty, the annihilation-operator, and the displacement-operator method.

The states obtained by extension of these three methods are in general different [4].

The displacement-operator coherent states are those preserving most of the properties of the harmonic-oscillator coherent states: they are still overcomplete, still enjoy resolution of unity, and moreover it was recently discovered [15] that they follow classical motion without dispersion by a dynamical feedback mechanism.

As to squeezed states, an extension of the minimum-uncertainty and annihilation-operator methods to arbitrary non harmonic systems was carried out by Nieto and collaborators [19], [20]. They also introduced an extension of the minimum-uncertainty method to obtain generalized coherent states [21].

However, an extension of the displacement-operator method to obtain generalized squeezed states runs into difficulties [22] and is still missing.

We now show that the generalized "dynamical" coherent states defined in the previous section via stochastic mechanics do in fact define a particular class of displacement-operator generalized squeezed states.

This is achieved by a natural extension of the strategy that allowed to connect the generalized coherent states introduced in Ref.[14] via stochastic mechanics with the displacement-operator generalized coherent states [15].

We proceed as follows. We first recall Glauber displacement operator written in the coordinate representation:

$$\hat{D}_{\alpha} \; = \; \exp\left(iS_0(t)\right) \exp\left(\frac{i}{\hbar}\langle\hat{p}\rangle\hat{q}\right) \exp\left(-\frac{i}{\hbar}\langle\hat{q}\rangle\hat{p}\right) \, . \tag{26}$$

This operator, when applied to any wave function $\Psi(x,t)$ displaces its space argument x into $x - \langle\hat{q}\rangle$ and adds to its phase the term $S_0(t) + \langle\hat{p}\rangle x$.

Next, we introduce a dynamical rescaling operator $\hat{S}_{\Delta q}$, namely a squeeze operator defined as

$$\hat{S}_{\Delta q} \; = \; \exp\left[i\left(\frac{f(t)}{\hbar}\{\hat{q},\hat{p}\} + \frac{g(t)}{(\Delta q_0)^2}\hat{q}^2\right)\right] \, , \tag{27}$$

where Δq_0 denotes the (time-independent!) ground state dispersion. Given $\Delta q(t)$ solution of Eq. (22), the two functions $f(t)$ and $g(t)$ read

$$f(t) \; = \; -\frac{1}{2}\ln\left(\frac{\Delta q}{\Delta q_0}\right) \, , \qquad g(t) \; = \; \frac{m}{\hbar}(1-2f(t))^{-1}\frac{d}{dt}\ln\Delta q \, . \tag{28}$$

We see from these relations that the function $f(t)$ plays the role of a dynamical squeezing parameter.

Next, it is easily shown that any ground state wave function can be cast in the general form

$$\Psi_0(x) \; = \; \frac{1}{\sqrt{\Delta q_0}} F\left(\frac{x}{\Delta q_0}\right) \, , \tag{29}$$

with F a suitably chosen function.

We now let $\hat{S}_{\Delta q}$ act on Ψ_0 to define the dynamically rescaled wave function

$$\Psi_{resc}(x,t) \;=\; \hat{S}_{\Delta q} \cdot \Psi_0(x)\,. \tag{30}$$

Exploiting Trotter product formula and the algebra of commutators, and observing that $\{\hat{q},\hat{p}\} = i\hbar(1+2xd/dx)$ and that $[\{\hat{q},\hat{p}\},\hat{q}^2] = -4i\hbar\hat{q}^2$, one obtains

$$\Psi_{resc}(x,t) \;=\; \exp\left[f(t)\left(1+2x\frac{d}{dx}\right)\right]\cdot\chi(x,t)\,, \tag{31}$$

where $\chi(x,t)$ is given by

$$\chi(x,t) \;=\; \exp\left[i\frac{g(t)}{(\Delta q_0)^2}(1-2f(t))x^2\right]\Psi_0(x)\,. \tag{32}$$

We now exploit the extension to the real axis of the following relation always true for analytic functions (used in Ref. [23] in the context of q-oscillators and coherent states):

$$Q^{x\frac{d}{dx}}\left[W(x)\right] \;=\; W(Qx)\,, \tag{33}$$

with Q any real c-number and W any function analytic on the real axis. Letting $Q = \exp[2f(t)]$ and $W = \chi$ one finally is left with

$$\Psi_{resc}(x,t) \;=\; \exp\left[f(t)+i\frac{g(t)}{(\Delta q_0)^2}(1-2f(t))e^{4f(t)}x^2\right]\Psi_0\left(e^{2f(t)}x\right)\,. \tag{34}$$

We then define the squeezed state Ψ_{sq} by applying $\hat{D}_\alpha$ on Ψ_{resc}:

$$\Psi_{sq}(x,t) \;=\; \hat{D}_\alpha\left(\hat{S}_{\Delta q}\cdot\Psi_0(x)\right) \;=\; \hat{D}_\alpha\cdot\Psi_{resc}(x,t)\,. \tag{35}$$

By recalling Eqs. (28) it is then straightforward to show that Ψ_{sq} coincides with the wave function Eq. (24) for the "dynamical" coherent states with classical motion and controlled dynamics of the dispersion. We thus proved that they can also be obtained by a suitable extension of the displacement-operator method.

Concluding remarks. We have introduced a class of generalized coherent states with time-dependent dispersion in the framework of Nelson stochastic quantization.

The wave packets follow a classical evolution due to a dynamical feedback, and the evolution equation controlling the spreading of the wave packet is naturally coupled with the classical evolution equation for the wave packet centre.

We then showed that these states can be obtained through a particular extension of the displacement-operator method by defining a dynamical rescaling operator.

The consequent dilatations or contractions of the wave packet width are then shown to be controlled by a single adimensional squeezing parameter $f(t)$.

In this letter we outlined the general features of the method: applications to specific potentials of physical interest, as well as a more thorough discussion of all the technical details will be given elsewhere [18].

It might be worth noting that we have chosen for simplicity a ground state to generate squeezed states by the action of operators (26)-(27); it is however immediately seen that their application on any stationary state yields again generalized coherent states of the form (24).

Finally, we remark that Eq. (23) represents the "stochastic squeezing" condition satisfied by our states. Namely, it expresses the complementary time-dependence of the spreading Δq and of the osmotic velocity uncertainty Δu.

It is easily seen that in the canonic picture Equations (10), (13), and (23) imply $\Delta\hat{q}^2\Delta\hat{p}^2 = K + L^2(t)$, with $L(t) = m\Delta\hat{q}d(\Delta\hat{q})/dt$.

The reciprocal variation in time of $\Delta\hat{q}$ and $\Delta\hat{p}$ is then ruled by $\Delta\hat{q}$ itself, determined as the solution of Eq. (22) with the initial condition $\Delta\hat{q}_0$. In this way squeezing is introduced as a self-consistent prescription on the dynamical evolution of the wave packet spreading.

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